DYNAMICS OF VIBRATION OF A CANTILEVER UNDER LATERAL IMPACT OF AN ELASTIC LOAD—PART II (GENERAL THEORY)

B. B. BANERJEE

PHYSICS DEPARTMENT, UNIVERSITY COLLEGE OF ENGINEERING, BURLA-ORISSA

(*Received September* 11, 1965)

ABSTRACT In Part I, the dynamics of the problem has been worked out for the struck points at the free end and at the middle of the cantilever. In this paper the general case of an elastic load, impinging transversely on any point of the cantilever is rigorously dealt with. The displacement of the cantilever at any point (including the struck point) has been worked out, and a general expression for the same is given. The displacement comes in form of a series and is found to be directly proportional to the striking velocity. Further it involves terms containing easily measurable quantities. The dynamics of the problem is developed without assuming any nature of force between the load and the bar. The theory is perfectly general.

INTRODUCTION

Symbols used in this part have the same significances as given in part I. Operational method due to Heaviside is employed to work out the problem.

With the help of eqn (4.1) and (4.2) Banerjee, (1966) the displacement at any point of the cantilever can be obtained.

For any point $x(0 \leqslant x \leqslant a)$, *eqn.* (4.1) is used

$$y_1 = y_a \frac{\Delta_1(\sinh nx - \sin nx) + \Delta_2(\cosh nx - \cos nx)}{\Delta_0}$$

where the values of Δ_1, Δ_2, Δ_0, are given by equation (5.1), (5.2) and (5.3) respectively (Banerjee 1966).

and as per eqn (9.1) (Banerjee 1966) $\frac{y_a}{v_0} = \frac{F(D)}{F_1(D)}$. where $F(D)$ stands for D and

$$F_1(D) = D^2 - \frac{E_1 I}{m} n^3 \left(1 + \frac{m}{E_2} D^2\right) f(D). \qquad \dots (1)$$

Therefore

$$\frac{y_1}{v_0} = \frac{f_0(D)}{F_3(D)} \qquad \dots (2)$$

where

$$f_0(D) = D[(\sinh\gamma \sin k_2\gamma \quad \cosh\gamma\cos k_2\gamma - \sinh k_2\gamma \sin\gamma \cdot \cos\gamma\cosh k_2\gamma$$
$$- \cosh k_1\gamma - \cos k_1\gamma)(\sinh\gamma\, x/l - \sin\gamma\, x/l) + (\sinh\gamma\cos k_2\gamma + \cosh k_2\gamma\sin\gamma$$
$$+ \sin k_1\gamma + \sinh k_1\gamma - \sinh k_2\gamma\cos\gamma - \cosh\gamma\sin k_2\gamma)(\cosh\gamma\, x/l - \cos\gamma\, x/l)] \quad \ldots \quad (3)$$

and

$$F_3(D) = 2\left[D^2\omega + \frac{E_0 I}{m} n^3\left(1 + \frac{m}{E_2}D^2\right)p\right] \quad \ldots \quad (4)$$

where

$$p = 2(1 + \cosh\gamma\cos\gamma) \quad \ldots \quad (5.1)$$

and

$$\omega = [\sinh k_2\gamma\cos k_2\gamma + \cosh k_1\gamma \sin k_1\gamma + \cosh k_1\gamma\cosh k_2\gamma\sin\gamma - \cosh k_2\gamma\sin k_2\gamma$$
$$-\sinh k_1\gamma\cos k_1\gamma - \cos k_1\gamma\cos k_2\gamma\sinh\gamma] \quad \ldots \quad (5.2)$$

With the help of **Heaviside's** expansion theorem

$$\frac{y_1}{v_0} = \frac{f_0(0)}{F_3(0)} + \sum \frac{f_0(\alpha_s)}{\alpha_s \cdot F_3'(\alpha_s)} e^{\alpha_s t} \quad \ldots \quad (6)$$

Now putting $D = 0$, $f_0(0) = 0$, and $F_3(0) \neq 0$.

Thus

$$\frac{y_1}{v_0} = \sum \frac{f_0(\alpha_s)}{\alpha_s \cdot F_3'(\alpha_s)} e^{\alpha_s t}$$

where summation extends over all roots of $D = [\alpha_s]$, $(s = 1, 2, 3, \ldots, r)$

For roots of D from $F_3(D) = 0$, in eqn. (4), we have $F_3(D) = 0$, whence

$$\frac{p}{w} = \frac{\frac{m}{M}\gamma}{1 - \frac{E_1 I}{E_2}\cdot\frac{m}{M}\cdot\frac{\gamma^4}{l^3}} \quad \ldots \quad (7)$$

Eqn. (7), can be solved graphically by drawing two sets of curves represented by

$$\eta_1 = \frac{p}{\omega} \quad \ldots \quad (8.1)$$

and

$$\eta_2 = \frac{\frac{m}{M}\gamma}{1 - \frac{E_1 I}{E_2}\cdot\frac{m}{M}\cdot\frac{\gamma^4}{l^3}} \quad \ldots \quad (8.2)$$

As $\eta_2 \sim \gamma$ curve exists only in the positive direction, the different values of γ_s of equation (7), are given by the points of intersections of two sets of curves $\eta_1 \sim \gamma_s$, and $\eta_2 \sim \gamma_s$, lying entirely in positive quadrants. Thus γ assumes different sets of values for different struck points given by the values of k_1 and k_2 etc. in particular beam-load system.

Thus $$nl = \gamma_s,\ \gamma_s = \text{pure number},\ (s = 1, 2, 3, \dots r) \qquad \dots \quad (9)$$

$$D = [\,\alpha_s\,] = \pm i q_s, \qquad \dots \quad (9.2)$$

$$q_s = \gamma_s^2 \cdot \sqrt{\frac{E_1 1}{M l^3}}\,, \qquad \dots \quad (9.3)$$

We have for equation (6)

$$F_3'(\alpha_s) = 2\omega[F_1'(\alpha_s)] \qquad \dots \quad (10.1)$$

and $F_1(\alpha_s)$ is obtained from $F_1(D)$ in eqn. (1) and is given by

$$F_1'(\alpha_s) = \frac{\alpha_s}{2}\left\{\frac{1-\dfrac{3m}{E_2}\alpha_s^2}{1+\dfrac{m}{E_2}\alpha_s^2} + \gamma_s\,\frac{\cosh\gamma_s\sin\gamma_s - \sinh\gamma_s\cos\gamma_s}{1+\cosh\gamma_s\cos\gamma_s} - \gamma_s\cdot\frac{L}{\omega}\right\} \qquad \dots \quad (10.2)$$

where ω is given by eqn. (5.2) and

$$\begin{aligned} L = {} & 2[k_2\sinh k_2\gamma_s\sin k_2\gamma_s - k_1\sinh k_1\gamma_s\sin k_1\gamma_s] + \cosh\gamma_s\cos k_1\gamma\cos k_2\gamma_s \\ & - \cosh k_1\gamma_s\cosh k_2\gamma_s\cos\gamma_s - k_1[\sin k_1\gamma_s\cos k_2\gamma_s\sinh\gamma_s + \sin\gamma_s\ \sinh k_1\gamma_s \\ & \cosh k_2\gamma_s] - k_2[\sin k_2\gamma_s\cos k_1\gamma_s\sinh\gamma_s + \sin\gamma_s\sinh k_2\gamma_s\cosh k_1\gamma]_s \end{aligned} \qquad \dots \quad (10.3)$$

Further

$$\sum \frac{e^{\alpha_s t}}{\alpha_s} = \sum \frac{2\sin q_s t}{q_s} \qquad \dots \quad (10.4)$$

Thus we have after simplification

$$y_1 = 2v_0 \sum_{s=1}^{r} \frac{\psi_1(\gamma_s)}{\psi(\gamma_s)} \cdot \frac{\sin q_s t}{q_s} \qquad \dots \quad (11)$$

where $\psi(\gamma_s) = \dfrac{2F_1'(\alpha_s)}{\alpha_s}$, where $F_1'(\alpha_s)$ is given by eqn. (10.2) with α_s^2 being replaced by $-q_s^2$

and

$$\sinh\gamma_s \sin k_2\gamma_s - \cosh\gamma_s \cos k_2\gamma_s - \sinh k_2\gamma_s \sin\gamma_s - \cosh k_2\gamma_s \cos\gamma_s$$

$$-\cosh k_1\gamma_s - \cos k_1\gamma_s)(\sinh\gamma_s x/l - \sin\gamma_s x/l) + (\sinh\gamma_s \cos k_2\gamma_s$$

$$+\cosh k_2\gamma_s \sin\gamma_s + \sin k_1\gamma_s + \sinh k_1\gamma_s - \sinh k_2\gamma_s \cos\gamma_s - \cosh\gamma_s$$

$$\psi_1(\gamma_s) = \frac{\sin k_2\gamma_s)\,(\cosh\gamma_s x/l - \cos\gamma_s x/l)}{w}$$

where w is given by (5.2) ... (11.1)

Equation (11) for y_1 can be represented in a convenient form as

$$y_1 = 2v_0 \sum \frac{c_s}{q_s} \sin q_s t \qquad \text{... (11.2)}$$

where c_s can be obtained from the quotient $\frac{\psi_1(\gamma_s)}{\psi(\gamma_s)}$ for $s = 1, 2, 3, \ldots r$.

When the observed point at x is the struck point itself, i.e. for any point, $x = a$, We have for equation (7), $f_0(0)$ and

$$F_1(0) = \frac{3E_1 I}{ma^3},$$

putting $D = 0$, as usual

$$\frac{f_0(0)}{F_3(0)} = 0, \text{ in eqn (6)}$$

and we get from eqn. (11)

$$y_a = 4v_0 \sum \frac{A_s}{q_s} \sin q_s t$$

where

$$\frac{1}{A_s} = \frac{1 + \frac{3m}{E_2} q_s^2}{1 - \frac{m}{E_2} q_s^2} + \gamma_s \frac{\cosh\gamma_s \sin\gamma_s - \sinh\gamma_s \cos\gamma_s}{1 + \cosh\gamma_s \cos\gamma_s} - \gamma_s \frac{L}{w} \qquad (12.1)$$

where L is given by eqn. (10.3)

The velocity of the load during impact is given by

$$v_t = \frac{dy_a}{dt} = 4v_0 \sum A_s \cos q_s t \qquad (13)$$

For hard load since E_2 is taken to be infinity, the $\eta_2 \sim \gamma$ relation as per eqn. (8.2) is written as

$$\eta_2 = \frac{m}{M} \gamma$$

and the $\eta_2 \sim \gamma$ curve is a straight line passing through the origin having an inclination such that

$$\tan \theta = \frac{m}{M} = \frac{\text{mass of the load}}{\text{mass of the bar}}$$

From eqn. (11), it is noted that the displacement of the cantilever is expressed in form of a series and is directly proportional to the impinging velocity of the load and since it contains γ, the displacement depends on the striking distance, measured from the fixed end, 'mass ratio', Young's modulus, length, and shape of the bar as also on the and elastic constant of the load. Different terms of the series, given by eqn. (11), represents successive modes of vibration, whose periods are obtained from q_s of eqn (9.3) for $s = 1, 2, 3, \ldots r$. Further from eqn. (11) it is found that the influence of different modes of vibration in developing the form of the displacement-time curve is not same throughout the length of the bar as it depends on γ_s.

For the struck point at the free end as also at the middle of the cantilever, i.e. for $x = a = l$, and for $x = a = l/2$, respectively, eqn. (11) which is the general form reduces to eqn. (13.1) and (13.2) (Banerjee, 66)

For any point $x(a \leqslant x \leqslant l)$, the displacement of the bar can be obtained from eqn. (4.2) (Banerjee, 66)

$$y_2 = y_a \frac{\Delta_3[\sinh n(l-x)+\sin n(l-x)]+\Delta_4[\cosh n(l-x)+\cos n(l-x)]}{}$$

where Δ_3, Δ_4, Δ_0, are respectively given by eqn. (5.3), (5.4) and (5.5) (Banerjee, 66) respectively.

Proceeding in the similar way as was followed in case of determination of y_x, we get the same roots of γ_s as given by eqn. (7)

Finally after simplification

$$y_2 = 2v_0 \sum_{s=1}^{r} \frac{\psi_2(\gamma_s)}{\psi(\gamma_s)} \frac{\sin q_s t}{q_s} \quad \ldots \quad (14)$$

where $\psi(\gamma_s)$ is given by $\frac{2F_1'(\alpha_s)}{\sim}$ where $F_1'(\alpha_s)$ is given by (10.2) and α_s^2 in eqn. (10.2) is written as $-q_s^2$

and

$$\psi_2(\gamma_s) = \frac{\begin{array}{l}(\cos k_2\gamma_s - \cosh k_2\gamma_s - \sinh \gamma_s \sin k_1\gamma_s - \sinh k_1\gamma_s \sin \gamma_s + \cosh \gamma_s \cos k_1\gamma_s \\ -\cosh k_1\gamma_s \cos \gamma_s)\left\{ \sinh \gamma_s\left(1-\frac{x}{l}\right)+\sin \gamma_s\left(1-\frac{x}{l}\right)\right\}+(\sinh k_2\gamma_s + \\ \cosh \gamma_s \sin k_1\gamma_s + \cosh k_1\gamma_s \sin \gamma_s - \sin k_2\gamma_s - \sinh \gamma_s \cos k_1\gamma_s \\ -\sinh k_1\gamma_s \cos \gamma_s)\left\{\cosh \gamma_s\left(1-\frac{x}{l}\right)+\cos \gamma_s\left(1-\frac{x}{l}\right)\right\}\end{array}}{\omega}$$

where ω is given by (5.2)

It should be noted here that E_2 which occurs in different equations is different from Young's modulus of the material of the load but it depends on the material size and shape of the hammer as also on the area of contact surface.

ACKNOWLEDGEMENT

My best thanks are due to Prof. M. M. Ghosh, D.Sc., vice-principal, City College, Calcutta for his lively interest in this work.

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